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CANONICAL QUANTUM ELECTRODYNAMICS IN COVARIANT GAUGES

BY

B. LAUTRUP

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Synopsis

A consistent canonical quantization procedure for electrodynamics in covariant gauges of a certain type is developed. This type comprises most of the gauges that usually are studied in the literature. In every gauge there are four photons and in the sense that the expectation value of the four-divergence of the Maxwell field is zero for all physical states, all these gauges are quantum generalizations of the classical Lorentz gauge. The quantization is carried out by means of a Lagrange multiplier field. It is shown that there exist generators for four-dimensional translations and rotations in every gauge. A peculiar aspect is that the scalar and longitudinal photons are not stationary states (except in one gauge), because the energy is not diagonalizable in general. This is connected with the necessity of introducing an indefinite metric. It is possible to connect the different gauges by operator "phase"-transformations of the electron field. The necessity of a gauge renormalization removes some difficulties with the usual formulation of quantum electrodynamics. The self-mass of the electron comes out gauge dependent by a direct calculation, but a more refined analysis shows that it actually is gauge independent.

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1. Introduction

In classical electrodynamics the free Lagrangian density of the electromagnetic potentials A_μ is

$$\mathfrak{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and we use the metric $g_{\mu\nu} = (1, -1, -1, -1)$. Variation of A_μ gives the classical equations of motion

$$\Box A_\mu + \partial_\mu \partial_\nu A^\nu = 0. \tag{2}$$

The freedom of gauge transformations allows us to choose both covariant and non-covariant gauge conditions, with the sole restriction that $F_{\mu\nu}$ be a covariant tensor. A typical example of a non-covariant gauge condition is div $\bar{A} = 0$ (the Coulomb gauge), and typical for the covariant gauges is the condition $\partial_\mu A^\mu = 0$ (the Lorentz gauge).[1]

It has been known for more than thirty-five years[2] that difficulties are met if we attempt to quantize (2) with a covariant gauge condition, e. g. the Lorentz condition. With modern methods we can see that such difficulties arise in every covariant gauge, in other words, that the equations (2) cannot be valid in any covariant gauge. From general arguments of field theory we find that the most general vacuum representation of the field commutator must be

$$\langle o | [A_\mu(x), A_\nu(y)] | o \rangle = -2\pi \int \vec{d}p\, \varepsilon(p) (\varrho_1(p^2) g_{\mu\nu} + \varrho_2(p^2) p_\mu p_\nu) e^{-ip\cdot(x-y)}, \tag{3}$$

where $\vec{d}p = dp(2\pi)^{-4}$ and $\varrho_1(p^2)$ and $\varrho_2(p^2)$ are spectral functions. Using (2) we find $\varrho_1 = 0$ so that it follows that

$$\langle o | [F_{\mu\nu}(x), F_{\varrho\sigma}(y)] | o \rangle = 0.$$

[1] As gauge transformations and Lorentz transformations may be mixed without disturbing the tensorial character of $F_{\mu\nu}$, the phrase non-covariant is somewhat ambiguous in this connexion. Consider for instance the Coulomb gauge. If we claim that the Coulomb condition has to hold in every inertial system, then A_μ does not transform like a four-vector, but according to a combined Lorentz and gauge transformation. If we claim that A_μ is a four-vector, then the Coulomb condition is not valid in every inertial system. For the covariant gauges no such difficulty arises because it is then natural to take A_μ to be a four-vector.

[2] W. Heisenberg and W. Pauli, Zeits. f. Physik **56** (1929) 1.

Now it is generally believed that this is not true[1], so that we must give up the equations (2) in covariant gauges. In a non-covariant gauge we cannot show that the vacuum representation has the form (3) and hence we cannot prove an analogous result in this case. The Maxwell equations (2) may very well be the equations of motion for the electromagnetic field in a non-covariant gauge. Among the non-covariant gauges we can mention the Coulomb gauge[2], the axial gauge[3], and the Valatin gauge[4].

In the case of the covariant gauges the above mentioned difficulty is usually overcome by adding a term

$$-\tfrac{1}{2}(\partial_\mu A^\mu)^2$$

to the Lagrangian density (1). The "gauge" obtained in this way is called the Fermi gauge, and it is the only covariant gauge which has been formulated as a theory of canonically quantized fields[5]. A price must, however, be paid for the simplicity obtained by this trick. There will be states which are not physically realizable, in the sense that they cannot be prepared in any experiment. Some arguments may be given to show that this is probably the case in every covariant gauge. Whereas we are not able to claim that the Maxwell equations (2) are satisfied for the quantized potentials, it seems very reasonable to claim that the mean value of the potentials should be real and satisfy

$$\Box\langle A_\mu\rangle + \partial_\mu\partial_\nu\langle A^\nu\rangle = 0 \tag{4}$$

in every physically realizable state[6]. But then all states cannot be physically realizable, because this would lead us back to (2).

In quantum electrodynamics most of the gauges which usually are studied belong to a one-parameter family characterized by the photon propagator

$$D_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2+i\varepsilon} + i(1-a)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}. \tag{5}$$

[1] R. E. Peierls, Proc. Roy. Soc. A **214** (1952) 143.
[2] L. E. Evans and T. Fulton, Nucl. Phys. **21** (1960) 492.
[3] R. L. Arnowitt and S. I. Fickler, Phys. Rev. **127** (1962) 1821.
J. Schwinger, Phys. Rev. **130** (1963) 402.
York-Peng Yao, Journ. Math. Phys. **5** (1964) 1319.
[4] J. G. Valatin, Mat. Fys. Medd. Dan. Vid. Selsk. **26** (1951) No. 13.
[5] See f. inst. G. Källén, Handbuch d. Phys. V_1 (Springer-Verlag, Berlin 1958).
[6] This is in analogy with the Ehrenfest theorem of non-relativistic quantum mechanics.

For $a = 1$ we get the Fermi gauge,
for $a = 0$ the Landau gauge[1] and
for $a = 3$ the Yennie gauge[2].

We shall show in this paper how the canonical quantization of a certain class of covariant gauges may be carried out in a systematic way. In this class we shall verify eq. (4) for the physical states. Furthermore, we shall for these gauges find

$$\partial_\mu \langle A^\mu \rangle = 0 \tag{6}$$

in every physical state, so that these gauges may all be considered as quantum generalizations of the classical Lorentz gauge. It will be shown that this class of gauges is essentially equivalent to the family given by eq. (5).

2. Quantization of the free Maxwell field

Let us begin with the study of quantum electrodynamics in the analogue of the classical Lorentz gauge, where

$$\partial_\mu A^\mu = 0 \tag{7}$$

is valid as an operator identity. Considering $A_\mu(x)$ to be generalized coordinates, we immediately see that (7) is a non-integrable relation between the generalized coordinates and velocities. This implies that quantum electrodynamics in this gauge is *non-holonomic* and hence the canonical methods cannot be expected to work at all. In classical mechanics non-holonomic systems with constraints like (7) are treated by means of Lagrange multipliers[3], and it is therefore tempting to use the same method here. Accordingly we add to the Lagrangian density (1) a term

$$-\Lambda \partial_\mu A^\mu,$$

where Λ is the Lagrange multiplier, which in this case must be a scalar field. In the new Lagrangian density

$$\mathfrak{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \Lambda \partial_\mu A^\mu \tag{8}$$

we are allowed to treat A_μ as independent coordinates, and it is immediately seen that the momentum canonically conjugate to A_0 is now $-\Lambda$,

[1] L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk. SSSR **95** (1954) 773; JETP **2** (1956) 69.
[2] H. M. Fried and D. R. Yennie, Phys. Rev. **112** (1958) 1391.
[3] R. Goldstein, Classical Mechanics, pp. 11, 40 (Addison-Wesley, 1959).

whereas it formerly was identically zero. It would now be possible to go on with the canonical quantization, but it is, however, convenient first to generalize (8) slightly. As variation of (8) after $\Lambda(x)$ gives us the gauge condition (7), it is seen that the Lagrange multiplier behaves like a free coordinate, with the constraint as "equation of motion". We therefore propose to change the Lagrangian density to

$$L = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \Lambda \partial_\mu A^\mu + F(\Lambda(x)) \tag{9}$$

where $F(z)$ is a holomorphic function with $F(o) = F'(o) = 0$. As this extra term not contains A_μ, it will only change the gauge condition to

$$\partial_\mu A^\mu(x) = F'(\Lambda(x)) \tag{10}$$

and should then only correspond to a gauge transformation. The equation of motion for A_μ is—independently of the function F—

$$\Box A_\mu + \partial_\mu \partial_\nu A^\nu = \partial_\mu \Lambda, \tag{11}$$

and from this we get the equation of motion for Λ

$$\Box \Lambda = 0. \tag{12}$$

From (9) we now find the momenta canonically conjugate to A_μ to be $\pi^{\mu o}$, where

$$\pi^{\mu\nu} = F^{\mu\nu} - g^{\mu\nu} \Lambda. \tag{13}$$

Canonical quantization leads to the relations

$$[A_\mu(x),\ A_\nu(y)]_{x_o = y_o} = 0, \tag{14}$$

$$[\dot{A}_i(x),\ A_\nu(y)]_{x_o = y_o} = i g_{i\nu} \delta(\bar{x} - \bar{y}), \tag{15}$$

$$[\Lambda(x),\ A_\nu(y)]_{x_o = y_o} = i g_{o\nu} \delta(\bar{x} - \bar{y}), \tag{16}$$

$$[\dot{A}_i(x),\ \dot{A}_k(y)]_{x_o = y_o} = 0, \tag{17}$$

$$[\Lambda(x),\ \dot{A}_k(y)]_{x_o = y_o} = -i \partial_k^x \delta(\bar{x} - \bar{y}), \tag{18}$$

$$[\Lambda(x),\ \Lambda(y)]_{x_o = y_o} = 0, \tag{19}$$

where the dot means differentiation with respect to time.

From (11) we find taking $\mu = 0$

$$\dot{\Lambda} = \Delta A_o + \partial_i \dot{A} \tag{20}$$

and then by (16) and (18)

$$[\dot{A}(x),\, A(y)]_{x_0=y_0} = 0\,.$$

Integration of (12) yields for arbitrary y_0

$$A(x) = -\int d\bar{y} D(x-y) \overleftrightarrow{\partial}_{y_0} A(y),$$

where $D(x-y)$ is the well-known singular function corresponding to mass zero[1]. We furthermore use the convention $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$. Hence we find that

$$[A(x),\, A(y)] = 0 \tag{21}$$

for arbitrary points x and y. Analogously we find by means of the equal-time commutation relations

$$[A(x),\, A_\mu(y)] = -\, i\partial_\mu^x D(x-y)\,. \tag{22}$$

Remark that the two important relations (21) and (22) both are independent of the gauge condition (10)[2]. It is possible to find one more set of relations which is independent of the gauge condition, namely the commutation relations for the field strengths $F_{\mu\nu}$. To find these we first define the transverse projection operator

$$\tau_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu + \frac{(\partial_\mu - n_\mu n\cdot\partial)\,(\partial_\nu - n_\nu n\cdot\partial)}{\Box + (n\cdot\partial)^2}, \tag{23}$$

where n_μ is a time-like unit vector. Here we shall always take $n_\mu = g_{\mu o}$. Then $\tau_{\mu\nu}$ only involves spatial operations[3]. Now define the transverse field by

$$A_\mu^{Tr} = \tau_{\mu\nu} A^\nu\,. \tag{24}$$

From (11) and the well-known properties of $\tau_{\mu\nu}$ it follows that

$$\Box A_\mu^{Tr} = 0\,. \tag{25}$$

Integrating (25) and using the equal time commutation relations we find

$$[A_\mu^{Tr}(x),\, A_\nu^{Tr}(y)] = -\, i\tau_{\mu\nu} D(x-y), \tag{26}$$

showing that the transverse part of the Maxwell field is correctly quantized. From (20) and (23) we get[3]

[1] G. Källén, ibid., p. 190.

[2] Equations (21) and (22) have been derived in the Fermi gauge of asymptotic quantum electrodynamics by R. E. Pugh (Ann. Phys. **30** (1964) 422).

[3] $\Delta = \Box + (n\cdot\delta)^2$ is Laplace's operator, and $1/\Delta$ may be defined as

$$\frac{1}{\Delta} f(\bar{x}) = -\int dx' \frac{f(\bar{x}')}{4\pi|\bar{x}-\bar{x}'|}$$

$$A_\mu = A_\mu^{Tr} + \frac{n_\mu}{\Delta}\dot{\Lambda} - \frac{\partial_\mu}{\Delta}\partial_i A^i$$

such that

$$F_{\mu\nu} = F_{\mu\nu}^{Tr} + \frac{\partial_\mu n_\nu - \partial_\nu n_\mu}{\Delta}\dot{\Lambda},$$

where $F_{\mu\nu}^{Tr}$ are the "transverse field strengths". Then using

$$[\Lambda(x), A_\mu^{Tr}(y)] = 0,$$

which follows from (22), we find

$$[F_{\mu\nu}(x), F_{\varrho\sigma}(y)] = i(g_{\mu\varrho}\partial_\nu\partial_\sigma - g_{\nu\varrho}\partial_\mu\partial_\sigma + g_{\nu\sigma}\partial_\mu\partial_\varrho - g_{\mu\sigma}\partial_\nu\partial_\varrho)D(x-y). \quad (27)$$

This is the most important result of this section. The quantization by means of a Lagrange multiplier method leads to the well-known commutation relations[1] for the field strengths in an arbitrary gauge of the type considered here.

In order to find the commutation relations for the potentials we shall make a special choice of gauge condition, namely

$$F(\Lambda(x)) = \frac{a}{2}\Lambda(x)^2, \quad (28)$$

where a is a real number, such that the gauge condition now reads

$$\partial_\mu A^\mu = a\Lambda. \quad (29)$$

In the appendix it is shown that this choice of gauge leads to the family of photon propagators (5). The gauge parameter a is actually identical to the mass ratio parameter in the theory of massive electrodynamics, developed by FELDMAN and MATTHEWS[2].

With the gauge condition (29) we find from (11)

$$\Box A_\mu = (1-a)\partial_\mu\Lambda. \quad (30)$$

Using (12) we get

$$\Box\Box A_\mu = 0.$$

Now it is fairly trivial to show that if a field $\varphi(x)$ satisfies the fourth order differential equation $\Box\Box\varphi(x) = 0$, then, for arbitrary y_o,

[1] R. E. PEIERLS, Proc. Roy. Soc. A **214** (1952) 143.
[2] G. FELDMAN and P. T. MATTHEWS, Phys. Rev. **130** (1963), 1633.

$$\varphi(x) = -\int d\bar{y} D(x-y)\overleftrightarrow{\partial}_{y_0}\varphi(y) - \int d\bar{y} E(x-y)\overleftrightarrow{\partial}_{y_0}\Box\varphi(y),$$

where $E(x)$ is given by

$$E(x) = 2\pi i \int đp\varepsilon(p)\delta'(p^2)e^{-ip(x-y)} = \left.\frac{\partial\Delta(x-y,\mu^2)}{\partial\mu^2}\right|_{\mu^2=0}.$$

The rather peculiar properties of this distribution are given in the appendix.

By means of the equal-time commutation relations and the integrated equation of motion for $A_\mu(x)$ we find after some calculation that

$$\left.\begin{aligned}[A_\mu(x), A_\nu(y)] &= -i(g_{\mu\nu}\Box + (1-a)\partial_\mu\partial_\nu)E(x-y) = \\ &= -ig_{\mu\nu}D(x-y) - i(1-a)\partial_\mu\partial_\nu E(x-y),\end{aligned}\right\} \quad (31)$$

where in the last line we have used the relation $\Box E(x) = D(x)$.

3. Indefinite metric

It is well known that it is necessary to introduce an indefinite metric in the Fermi gauge in order to secure the covariance of certain expression[1]. As the present theory contains the Fermi gauge as a special case, it must be expected that this will also be necessary in any covariant gauge of the type considered here.

For the moment we shall content ourselves with the following properties of the metric operator η:

$$\eta = \eta^* = \eta^{-1}, \quad (32)$$

$$\eta A_\mu^* \eta = A_\mu, \quad (33)$$

$$\eta|o> = |o>. \quad (34)$$

Equation (32) expresses the Hermiticity and unitarity of the metric operator, (33) the self-adjointness properties of A_μ with respect to η, and (34) the choice of positive norm for the vacuum. In the following section we shall fix the properties of η with respect to A_μ completely.

[1] G. Källén ibid., p. 191, 199. Here further references can be found.

4. Fourier expansion of the field

In this section we limit ourselves to the case (29). Let us define the field

$$\chi(x) = \frac{a-1}{2}\frac{1}{\Delta}\left(x_o\dot{\Lambda}(x) - \tfrac{1}{2}\Lambda(x)\right). \tag{35}$$

From eq. (12) it then follows that

$$\Box\chi(x) = (1-a)\Lambda(x).$$

This shows that the field

$$A_\mu^F = A_\mu - \partial_\mu\chi(x) \tag{36}$$

satisfies the equation

$$\partial^\mu A_\mu^F = \Lambda, \tag{37}$$

and from (11) we find the equations of motion for A_μ^F:

$$\Box A_\mu^F = 0. \tag{38}$$

From the relation (see the appendix)

$$E(x) = \frac{1}{2\Delta}\left(D(x) - x_o\dot{D}(x)\right)$$

we get using eqs. (21), (22), (31), (35), and (36)

$$[A_\mu^F(x), A_\nu^F(y)] = -ig_{\mu\nu}D(x-y). \tag{39}$$

As eqs. (38) and (39) are the equations of motion and commutation relations of the Maxwell field in the Fermi gauge we may immediately write down the usual expansion (in a periodicity volume)

$$A_\mu^F(x) = \frac{1}{\sqrt{V}}\sum_{\bar{k}\lambda}\frac{e_\mu(\bar{k}\lambda)}{\sqrt{2\omega}}\left(a(\bar{k}\lambda)e^{-ik\cdot x} + \tilde{a}(\bar{k}\lambda)e^{ik\cdot x}\right), \tag{40}$$

where $a(\bar{k}\lambda)$ and

$$\tilde{a}(\bar{k}\lambda) = \eta a^*(\bar{k}\lambda)\eta$$

have the usual commutation properties

$$[a(\bar{k}\lambda), \tilde{a}(\bar{k}'\lambda')] = \delta_{\bar{k}\bar{k}'}(-g_{\lambda\lambda'}),$$
$$[a(\bar{k}\lambda), a(\bar{k}'\lambda')] = 0.$$

If we therefore fix the properties of η by the relations

$$[\eta, a(\bar{k}\lambda)] = 0 \quad (\lambda = 1, 2, 3),$$
$$\{\eta, a(\bar{k}o)\} = 0,$$

we find

$$[a(\bar{k}\lambda), a^*(\bar{k}'\lambda')] = \delta_{\bar{k}\bar{k}'}\,\delta_{\lambda\lambda'}. \tag{41}$$

This permits us to interpret $a(\bar{k}\lambda)$ and $a^*(\bar{k}\lambda)$ as annihilation and creation operators for photons.

By virtue of equation (12) we may expand the Λ-field as

$$\Lambda(x) = \frac{i}{\sqrt{V}} \sum_{\bar{k}} \frac{\omega}{\sqrt{2\omega}} (\lambda(\bar{k}) e^{-ik\cdot x} - \tilde{\lambda}(\bar{k}) e^{ik\cdot x}), \tag{42}$$

where as above $\tilde{\lambda}(\bar{k}) = \eta \lambda^*(\bar{k}) \eta$. From eqs. (37) and (40) we then find

$$\lambda(\bar{k}) = a(\bar{k}3) - a(\bar{k}o). \tag{43}$$

One may easily verify that

$$[\lambda(\bar{k}), \lambda(\bar{k}')] = [\lambda(\bar{k}), \tilde{\lambda}(\bar{k}')] = 0 \tag{44}$$

in accordance with the vanishing of the Λ-Λ commutator. Remark that (44) leans heavily on the properties of the indefinite metric.

By the expansion (42), of the Λ-field and by eqs. (35), (36), (40) and (43) we may now express the total field A_μ in terms of annihilation and creation operators.

As in every gauge we are able to define the creation and annihilation operators as above, the Hilbert (Fock) space will have the same structure in every gauge. Although electrodynamics in different covariant gauges must be considered as different field theories (because the gauge condition is stated before the derivation of the equations of motion), we can, however, think of these theories as formulated in the same Hilbert (Fock) space.

The transformation field $\chi(x)$, given in (35) seems to be non-covariant on account of the explicit time dependence. [If the equations of motion for A_μ are solved by Fourier transformation one finds that this time dependence essentially stems from the term $-(t/2\omega)\cos\omega t$ in the solution of differential equations of the type

$$\frac{d^2y}{dt^2} + \omega^2 y = \sin \omega t].$$

The question of the apparent non-covariance is resolved in the following way. Covariance in field theory is equivalent to showing the existence of a representation of the proper inhomogeneous Lorentz group under which the fields transform correctly, i. e. finding generators of infinitesimal translations and rotations. As we shall show in section 5 these generators exist, but are not independent of the gauge. This means that the field $A_\mu^F(x)$, which can be defined in every gauge by eq. (36), is only a four-vector in the Fermi gauge. The splitting $A_\mu = A_\mu^F + \partial_\mu \chi$ is therefore a splitting of the covariant field A_μ into two non-covariant terms in the gauge characterized by the parameter value a. An expression which is covariant in one gauge need not be so in any other gauge.

5. The energy-momentum tensor

Although it would be possible to study the general gauge condition (10) we shall here limit ourselves to the simple case (29) where the Lagrangian density is

$$\mathfrak{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \Lambda \partial_\mu A^\mu + \frac{a}{2} \Lambda^2 . \tag{45}$$

By straightforward calculation we find from (45) the symmetric energy-momentum tensor

$$T_{\mu\nu} = -F_{\mu\lambda} F_\nu^{\cdot\lambda} + \tfrac{1}{4} g_{\mu\nu} F_{\varrho\sigma} F^{\varrho\sigma} + A_\mu \partial_\nu \Lambda + A_\nu \partial_\mu \Lambda - g_{\mu\nu} (A_\lambda \partial^\lambda \Lambda + \frac{a}{2} \Lambda^2), \tag{46}$$

which by means of the equations of motion and the gauge condition is seen to be conserved.

By direct calculation it is possible to show that the linear and angular momentum operators P_μ and $M_{\mu\nu}$ have the correct commutation properties with respect to the Maxwell field, i. e. that they are generators of infinitesimal translations and rotations. Remark, however, that P_μ and $M_{\mu\nu}$ are not Hermitian, but self-adjoint, i. e. $\eta P_\mu^* \eta = P_\mu$ and analogously for $M_{\mu\nu}$. This has the consequence that for instance the energy H is not necessarily diagonalizable. Take for instance the model (α is real)

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 1+\alpha & \alpha \\ -\alpha & 1-\alpha \end{pmatrix}. \tag{47}$$

This H is self-adjoint with respect to η, but it has only one eigenvector, for $\alpha \neq o$.

Using $A_\mu = A_\mu^F + \partial_\mu \chi$ we can split the energy-momentum tensor (46) into two parts

$$T_{\mu\nu} = T_{\mu\nu}^F + T_{\mu\nu}^\Lambda,$$

where the first part is what we would find in the Fermi gauge and the second is only dependent on Λ. From the usual theory of the Fermi gauge we know that (after removal of zeropoint terms)

$$P_\mu^F = \int d\bar{x}\, T_{\mu o}^F = \sum_{\bar{k}\lambda} k_\mu a^*(\bar{k}\lambda) a(\bar{k}\lambda).$$

The rest of the four-momentum

$$P_\mu^\Lambda = \int d\bar{x}\, T_{\mu o}^\Lambda$$

is now found to be

$$P_i^\Lambda = 0,$$

$$P_o^\Lambda = \int d\bar{x} \frac{1-a}{4}\left(\Lambda^2 - \dot{\Lambda}\frac{1}{\Delta}\dot{\Lambda}\right) = \frac{1-a}{2}\sum_{\bar{k}} \omega \tilde{\lambda}(\bar{k})\lambda(\bar{k}).$$

Finally, we have the total momentum and energy

$$P_i = \sum_{\bar{k}\lambda} k_i a^*(\bar{k}\lambda)\, a(\bar{k}\lambda),$$

$$P_o = H = \sum_{\bar{k}\lambda} \omega a^*(\bar{k}\lambda)\, a(\bar{k}\lambda) + \frac{1-a}{2}\sum_{\bar{k}} \omega \tilde{\lambda}(\bar{k})\lambda(\bar{k}).$$

From this it follows that (in matrix notation)

$$\left[H, \begin{pmatrix} a^*(\bar{k}3) \\ a^*(\bar{k}0) \end{pmatrix}\right] = \omega \begin{pmatrix} 1 + \frac{1-a}{2} & \frac{1-a}{2} \\ -\frac{1-a}{2} & 1 - \frac{1-a}{2} \end{pmatrix} \begin{pmatrix} a^*(\bar{k}3) \\ a^*(\bar{k}0) \end{pmatrix},$$

showing—by comparison with (47)—that the energy is not diagonalizable, except in the Fermi gauge ($a = 1$). This means that the longitudinal and scalar photons are not in general stationary states, but mix with each other during time. Only one combination of scalar and longitudinal photons is stationary, namely

$$a^*(\bar{k}3) + a^*(\bar{k}0) = \tilde{\lambda}(\bar{k}).$$

Let us by a natural generalization of the definition of physical states in the Fermi gauge demand that the physical states satisfy

$$\Lambda_+(x)|\Phi> = 0, \tag{48}$$

where Λ_+ is the positive frequency part of Λ. Then it is easy to see that only the transverse photons contribute to the mean value of the energy in a physical state. Furthermore

$$\langle \Phi | \partial_\mu A^\mu | \Phi \rangle = 0, \tag{6}$$

so that every gauge of the type studied here must be considered as a quantum generalization of the classical Lorentz gauge in the same sense as the Fermi gauge. (It is also seen that eq. (4) is satisfied.)

6. Quantization of the interacting fields

We shall now consider the Maxwell field in interaction with the electron field. The Lagrangian density is taken to be[1]

$$\mathfrak{L} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \Lambda\partial_\mu A^\mu + \frac{a_o}{2}\Lambda^2 + \tfrac{1}{2}[\overline{\psi}, (i\gamma\cdot\partial - m_o)\psi] - \frac{e_o}{2}[\overline{\psi}, \gamma_\mu\psi]A^\mu, \tag{49}$$

where as before we have treated the gauge condition by means of a Λ-field. All the fields are considered to be unrenormalized, and a_o, m_o, e_o are the unrenormalized parameters of the theory. By calling the gauge parameter a_o, we have admitted the possibility of a gauge renormalization, and we have restricted ourselves to the simple gauge conditions of the type (29). The equations of motion are found to be

$$\Box A_\mu + \partial_\mu\partial_\nu A^\nu = \partial_\mu\Lambda - \frac{e_o}{2}[\overline{\psi}, \gamma_\mu\psi], \tag{50}$$

$$\partial_\mu A^\mu = a_o\Lambda, \tag{51}$$

$$(i\gamma\cdot\partial - m_o)\psi = e_o\gamma\cdot A\psi, \tag{52}$$

from which we again derive

$$\Box\Lambda = 0. \tag{53}$$

[1] We use the γ-matrices with $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ and $\gamma\cdot a = a_\mu\gamma^\mu$. These γ-matrices are self-adjoint, i. e. $\overline{\gamma}_\mu = \gamma_o\gamma_\mu^+\gamma_o = \gamma_\mu$.

The metric operator η, which must also be introduced in this case, with the properties given in section 3, is moreover assumed to satisfy $\overline{\psi} = \eta\psi^{+}\eta\gamma_o$.

The equal-time commutation relations for the Maxwell field and the Λ-field are found to be exactly the same as in the free case (eqs. (14)–(19)). Furthermore we have the following commutation and anticommutation relations:

$$\{\psi(x),\ \psi(y)\}_{x_o = y_o} = 0\,, \tag{54}$$

$$\{\psi(x),\ \overline{\psi}(y)\}_{x_o = y_o} = \gamma_o \delta(\bar{x} - \bar{y})\,, \tag{55}$$

$$[\psi(x),\ A_\mu(y)]_{x_o = y_o} = [\overline{\psi}(x),\ A_\mu(y)]_{x_o = y_o} = 0\,, \tag{56}$$

$$[\psi(x),\ \dot{A}_i(y)]_{x_o = y_o} = [\overline{\psi}(x),\ \dot{A}_i(y)]_{x_o = y_o} = 0\,, \tag{57}$$

$$[\psi(x),\ \Lambda(y)]_{x_o = y_o} = [\overline{\psi}(x),\ \Lambda(y)]_{x_o = y_o} = 0\,, \tag{58}$$

By methods analogous to those used in section 2 we find

$$[\Lambda(x),\ \Lambda(y)] = 0\,, \tag{59}$$

$$[\Lambda(x),\ A_\mu(y)] = -\,i\partial_\mu^x D(x-y)\,, \tag{60}$$

$$[\Lambda(x),\ \psi(y)] = e_o D(x-y)\psi(y)\,, \tag{61}$$

$$[\Lambda(x),\ \overline{\psi}(y)] = -\,e_o D(x-y)\overline{\psi}(y)\,. \tag{62}$$

These are the only integrable commutation relations in the case of the interacting fields. A consequence of the last two relations is for instance that $\Lambda(x)$ commutes with any local bilinear expression in ψ and $\overline{\psi}$, i. e.

$$[\Lambda(x),\ \psi_\alpha(y)\overline{\psi}_\beta(y)] = [\Lambda(x),\ \overline{\psi}_\beta(y)\psi_\alpha(y)] = 0\,. \tag{63}$$

7. Gauge transformation

If we had started with another value for the gauge parameter, say a'_o, then we would have arrived at a different theory of interacting fields. In the case of the free Maxwell field we have however seen that the two gauges could be connected by a gauge transformation

$$A'_\mu = A_\mu + \partial_\mu \chi \tag{64}$$

with

$$\chi(x) = \frac{a_0' - a_0}{2} \frac{1}{\Delta}\left(x_0 \dot{\Lambda}(x) - \tfrac{1}{2}\Lambda(x)\right). \tag{65}$$

We shall now see that the transformation (64) with $\chi(x)$ given by (65) also carries us from the gauge a_o to the gauge a_o' in the case of interacting fields, when the electron field is subjected to the transformation

$$\psi'(x) = e^{-i\frac{e_0}{2}\chi(x)}\psi(x)e^{-i\frac{e_0}{2}\chi(x)}. \tag{66}$$

One should here remark that $\chi(x)$ is a q-number which does not commute with $\psi(x)$.

We shall now show that provided A_μ and ψ satisfy the equations of motion and commutation relations for the gauge a_0, then A_μ' and ψ' satisfy the equations of motion and commutation relations for the gauge a_0'.

From eqs. (50) and (64) we find

$$\Box A_\mu' + \partial_\mu \partial_\nu A'^\nu = \partial_\mu \Lambda - \frac{e_0}{2}[\overline{\psi}, \gamma_\mu \psi].$$

However, by (63) we have

$$[\overline{\psi}', \gamma_\mu \psi'] = [\overline{\psi}, \gamma_\mu \psi].$$

Likewise by (51), (53), (64), (65) we find

$$\partial_\mu A'^\mu = a_0' \Lambda.$$

From (52) and (66) we get

$$(i\gamma\cdot\partial - m_0)\psi' = e_0 \gamma\cdot A'\psi' - i\frac{e_0^2}{2}[\chi, \gamma\cdot A]\psi' - \frac{e_0}{2}[\gamma\cdot\partial\chi, \psi'], \tag{67}$$

where we have used the fact that $[\chi(x), A_\mu(y)]$ is a c-number such that

$$e^{-i\frac{e_0}{2}\chi}A_\mu e^{+i\frac{e_0}{2}\chi} = A_\mu - \frac{ie_0}{2}[\chi, A_\mu].$$

Unless a cancellation between the singular terms on the right hand side of eq. (67) occurs, we cannot reduce it to the required form. But as we shall soon see, this cancellation actually occurs. On account of the singular character of the terms we have to treat them with some care. First we find by means of the commutation relations (61) that

$$[\gamma\cdot\partial\,\chi(x),\,\psi(y)]_{x_0=y_0} = -\frac{a_0'-a_0}{2\,\Delta_x}\,e_0\{\tfrac{1}{2}\gamma_0+x_0\gamma_i\partial_x^i\}\,\delta(\bar{x}-\bar{y})\psi(\bar{y},x_0),$$

while from (60)

$$[\chi(y),\,\gamma\cdot A(x)]_{x_0=y_0} = -i\frac{a_0'-a_0}{2\,\Delta_x}(\tfrac{1}{2}\gamma_0+x_0\gamma_i\partial_x^i)\delta(\bar{x}-\bar{y}).$$

Then

$$ie_0[\chi(y),\,\gamma\cdot A(x)]_{x_0=y_0}\psi(\bar{y},x_0)+[\gamma\cdot\partial\,\chi(x),\,\psi(y)]_{x_0=y_0} = 0.$$

Here we can take $\bar{x}=\bar{y}$ and by use of (66) we see that the last two terms in (67) actually cancel each other.

It is possible to show that the commutation relations are invariant under these finite transformations, but it must actually suffice to show that they are invariant under infinitesimal transformations with $\delta a_0 = a_0' - a_0$ infinitesimal. As an example we take the anticommutator (55). Then

$$\delta\{\psi(x),\bar{\psi}(y)\}_{x_0=y_0} = -i\frac{e_0}{2}(\{\{\chi(x),\psi(x)\},\bar{\psi}(y)\}_{x_0=y_0}-\{\psi(x),\{\chi(y),\bar{\psi}(y)\}\}_{x_0=y_0}).$$

Using

$$\{A,\{B,C\}\} = [[A,B],C]+\{B,\{C,A\}\}$$

we find by means of the commutation relations

$$\delta\{\psi(x),\bar{\psi}(y)\}_{x_0=y_0} = -i\frac{e_0}{2}\bigg(-\frac{\delta a_0}{2\,\Delta_x}x_0e_0\delta(\bar{x}-\bar{y})[\bar{\psi}(y),\psi(x)]+$$

$$+2\,\chi(x)\gamma_0\delta(\bar{x}-\bar{y})-\frac{\delta a_0}{2\,\Delta_y}y_0e_0\delta(\bar{x}-\bar{y})[\psi(x),\bar{\psi}(y)]-2\,\chi(y)\gamma_0\delta(\bar{x}-\bar{y})\bigg)_{x_0=y_0} = 0.$$

The other commutation relations are shown to be invariant in an analogous manner.

It should be remarked that the Λ-field is not supposed to transform, in other words that it is assumed to be gauge invariant. This is also consistent with the manifestly gauge independent form of the left hand side of eq (50). *The Λ-field has more to do with the general mode of description (covariant gauges) than with the particular gauge in which this description is carried out.* Also the constants m_0 and e_0 must be supposed to be gauge independent.

8. Renormalization

A characteristic feature of the renormalization in the Fermi gauge is the occurrence of a term

$$-\frac{L}{1-L}\partial_\mu\partial_\nu A^\nu \tag{68}$$

in the renormalized current. This term cannot be accounted for by charge or wave function renormalization, but has to be introduced by gauge invariance arguments[1]. If the correction to the photon propagator is calculated in the lowest order in the unrenormalized theory, it is seen that starting from the Fermi gauge, one does not end up with a propagator which behaves near the mass shell like a propagator in the Fermi gauge. In the renormalized Fermi gauge the term (68) brings us back again to this form and hence the inclusion of this term must be considered as a renormalization of the gauge parameter. We shall show that this gauge renormalization comes out quite naturally in the present formulation.

Let us now introduce the renormalized mass (m), charge (e) and gauge parameter (a) by

$$m_0 = m - \delta m, \tag{69}$$

$$e_0 = \frac{e}{\sqrt{1-L}}, \tag{70}$$

$$a_0 = (1-K)a, \tag{71}$$

where δm, L, K are renormalization constants to be determined later on. Likewise we introduce the renormalized fields $\psi^{(r)}$, $A_\mu^{(r)}$ and $\Lambda^{(r)}$ by

$$\psi = N\psi^{(r)}, \tag{72}$$

$$A_\mu = \sqrt{1-L}\,A_\mu^{(r)}, \tag{73}$$

$$\Lambda = \frac{\sqrt{1-L}}{1-K}\Lambda^{(r)} \tag{74}$$

where we have anticipated the result that the wave function renormalization constant of the Maxwell field is the same as the charge renormalization constant. The wave function renormalization constant of the Λ-field has been chosen such that the gauge condition for the renormalized fields reads

$$\partial_\mu A^{(r)\mu} = a\Lambda^{(r)}. \tag{75}$$

[1] G. Källén, ibid., p. 346.

One might object that the Λ-field should not be renormalized as it does apparently not take part in the interaction on account of eq. (53). But a glance on eq. (61) shows that this is not true.

Leaving out the superscript (r) on the renormalized fields we now find the renormalized equations of motion

$$\Box A_{\mu} + \partial_{\mu}\partial_{\nu}A^{\nu} = \partial_{\mu}\Lambda - J_{\mu}, \tag{76}$$

$$J_{\mu} = \frac{eN^2}{2(1-L)}[\overline{\psi}, \gamma_{\mu}\psi] - \frac{K}{1-K}\partial_{\mu}\Lambda, \tag{77}$$

$$(i\gamma\cdot\partial - m)\psi = -f, \tag{78}$$

$$f = - e\gamma\cdot A\psi + \delta m\psi, \tag{79}$$

$$\partial_{\mu}A^{\mu} = a\Lambda. \tag{80}$$

It is seen that a term of the type (68) is now present in the renormalized current. We shall actually find in the next section that $K = L$. The commutation relations between the renormalized fields may easily be derived from the previously stated relations for the unrenormalized fields.

9. Determination of the renormalization constants

The renormalization constants are usually expressed by integrals over some spectral functions. By the well-known arguments we can write the vacuum expectation value of the current-current commutator as

$$\langle o|[J_{\mu}(x), J_{\nu}(y)]|o\rangle = -i(g_{\mu\nu}\Box + \partial_{\mu}\partial_{\nu})\int_{o}^{\infty} d\lambda \Delta(x-y,\lambda)\Pi(\lambda), \tag{81}$$

where $\Pi(\lambda)$ is a positive definite spectral function[1] (zero for negative λ) and

$$\Delta(x-y,\lambda) = -2\pi i\int dp\varepsilon(p)\delta(p^2-\lambda)e^{-ip(x-y)} \tag{82}$$

is the singular function with mass $\sqrt{\lambda}$.

In order to find the renormalization constants in terms of this spectral function the standard procedure is to integrate (81) with suitable limit conditions, which express how the interacting field asymptotically goes over

[1] As the definition (81) is obviously gauge invariant we may use the result from the Fermi gauge (G. Källén, ibid., p. 350).

into the incoming free field. Asymptotic conditions are, however, very difficult to apply, because the limits are not well-defined. Furthermore, one finds in the conventional theory of the Fermi gauge that the matrix element of the renormalized field between vacuum and a one-photon state is different from the same matrix element of the incoming free field, because[1]

$$\langle o | A_\mu(x) | k \rangle = (g_{\mu\nu} - M \partial_\mu \partial_\nu) \langle o | A^{(in)\nu}(x) | k \rangle \tag{83}$$

where M is a non-vanishing constant. Asymptotic conditions of the conventional form[2] can therefore not be applied to quantum electrodynamics.

It is, however, possible to integrate (81) without transitions to the limit of the infinite past or future. Let us define the field

$$A_\mu^{(y_0)}(x) = -\int d\bar{y} D(x-y) \overleftrightarrow{\partial}_{y_0} A_\mu(y). \tag{84}$$

By an elementary integration by parts we find

$$A_\mu(x) = A_\mu^{(y_0)}(x) + \int_{y_0}^{x_0} dx' D(x-x') \Box A_\mu(x'). \tag{85}$$

Now from the equal-time commutation relations we find

$$[A_\mu^{(y_0)}(x), J_\nu(y)] = i \frac{K}{1-L} \partial_\mu \partial_\nu D(x-y). \tag{86}$$

Then we have

$$[A_\mu(x), J_\nu(y)] = i \frac{K}{1-L} \partial_\mu \partial_\nu D(x-y) - \int_{y_0}^{x_0} dx' D(x-x') [J_\mu(x'), J_\nu(y)],$$

where we also have used the equations of motion and the fact that the A-field commutes with the current. If we take the vacuum expectation value of this equation and insert (81) we find after some calculation

$$\left.\begin{aligned} \langle o | [A_\mu(x), J_\nu(y)] | o \rangle &= i \frac{K}{1-L} \partial_\mu \partial_\nu D(x-y) + \\ + i(g_{\mu\nu} \Box + \partial_\mu \partial_\nu) & \int_0^\infty d\lambda (\Delta(x-y,\lambda) - D(x-y)) \frac{\Pi(\lambda)}{\lambda} - \\ - i(g_{\mu\nu} - g_{\mu o} g_{\nu o}) & D(x-y) \int_0^\infty d\lambda \, \Pi(\lambda), \end{aligned}\right\} \tag{87}$$

[1] G. Källén, ibid., p. 344.
[2] H. Lehmann, K. Zymanzik, W. Zimmermann, Nuovo Cimento 1 (1955) 425.

where we have used the relation

$$\int_{y_0}^{x_0} dx' D(x-x')\Delta(x'-y,\lambda) = \frac{1}{\lambda}\left(\Delta(x-y,\lambda) - D(x-y)\right). \quad (88)$$

The last term in (87) is non-covariant and must therefore be identically zero, i. e.

$$\int_{o}^{\infty} d\lambda \Pi(\lambda) = 0. \quad (89)$$

This also follows from the application of current conservation to (87). In view of the positive definite character of $\Pi(\lambda)$ this result seems quite nonsensical. A suitable regularization procedure, however, removes this difficulty[1], and without going into details we shall in the following assume eq. (89) to be valid.

By a similar procedure we can now integrate (87) to get

$$\left.\begin{aligned} &\langle o|[A_\mu(x), A_\nu(y)]|o\rangle = \\ &= -i\left[(g_{\mu\nu}\Box + \partial_\mu\partial_\nu)\left(\frac{1}{1-L} - \overline{\Pi}(o)\right) - a\frac{1-K}{1-L}\partial_\mu\partial_\nu\right] E(x-y) - \\ &- i(g_{\mu\nu}\Box + \partial_\mu\partial_\nu)\int_{o}^{\infty} d\lambda \Delta(x-y,\lambda)\left(\frac{\Pi(\lambda)}{\lambda^2} - \overline{\Pi}'(o)\delta(\lambda)\right), \end{aligned}\right\} \quad (90)$$

where

$$\overline{\Pi}(o) = \int_{o}^{\infty} d\lambda \frac{\Pi(\lambda)}{\lambda}, \quad (91)$$

$$\overline{\Pi}'(o) = \int_{o}^{\infty} d\lambda \frac{\Pi(\lambda)}{\lambda^2}, \quad (92)$$

and we have used the relation

$$\int_{y_0}^{x_0} dx' D(x-x') D(x'-y) = E(x-y).$$

which can be obtained either from (88) by letting $\lambda \to o$ or by direct calculation.

Our renormalization requirement is then that this commutator shall behave like the commutator for the free Maxwell field in the gauge a, near the mass shell. If we disregard the δ-function in the last term this gives us

[1] J. Moffat, Nucl. Phys. **16** (1960) 304.

$$\frac{1}{1-L} = 1 + \overline{\Pi}(o), \tag{93}$$

$$\frac{1-K}{1-L} = 1, \tag{94}$$

i. e. $K = L$.

A similar procedure may be carried out for the electron field, for which the spectral functions are defined by

$$\langle o | \{ f(x), \bar{f}(y) \} | o \rangle = -2\pi \int d\!\!\!^{-}p \varepsilon(p) (\Sigma_1(p^2) - (\gamma \cdot p - m) \Sigma_2(p^2)) e^{-ip\cdot(x-y)}. \tag{95}$$

The renormalization constants are then found to be given by

$$\delta m = N^2 \overline{\Sigma}_1(m^2), \tag{96}$$

$$\frac{1}{N^2} = 1 + \overline{\Sigma}_2(m^2) - 2\,\mathrm{m}\,\overline{\Sigma}_1'(m^2), \tag{97}$$

as in the conventional theory of the Fermi gauge. We have used the notation

$$\overline{\Sigma}_i(m^2) = \int_{m^2}^{\infty} d\lambda \frac{\Sigma_i(\lambda)}{\lambda - m^2}, \tag{98}$$

$$\overline{\Sigma}_1'(m^2) = \int_{m^2}^{\infty} d\lambda \frac{\Sigma_1(\lambda)}{(\lambda - m^2)^2}. \tag{99}$$

10. Asymptotic conditions for the Maxwell field

We now assume the existence of incoming free fields $A_\mu^{(in)}$, $\Lambda^{(in)}$, such that

$$\Box A_\mu^{(in)} + \partial_\mu \partial^\nu A_\nu^{(in)} = \partial_\mu \Lambda^{(in)}, \tag{100}$$

$$\partial^\mu A_\mu^{(in)} = a \Lambda^{(in)}, \tag{101}$$

$$[A_\mu^{(in)}(x), A_\nu^{(in)}(y)] = -i(g_{\mu\nu} \Box + (1-a)\partial_\mu \partial_\nu) E(x-y). \tag{102}$$

It is then possible to show that with a suitable value for M the following asymptotic conditions are consistent:

$$\Lambda = \Lambda^{(in)}, \tag{103}$$

$$A_\mu = A_\mu^{(in)} - M\partial_\mu \Lambda + \int dx' D_R(x-x') J_\mu(x'), \tag{104}$$

where D_R is the retarted photon Green function[1]. It is quite clear that these conditions are consistent with the equations of motion. But still we have to show that for a suitable M they will also be consistent with the spectral resolution of the commutator. From a spectral analysis of $\langle o|[A_\mu(x), A_\nu^{(in)}(y)]|o\rangle$ it follows by means of the equations of motion and (103) that

$$\langle o|[J_\mu(x), A_\nu^{(in)}(y)]|o\rangle = 0. \tag{105}$$

From this we find by means of (104) that

$$\langle o|A_\mu(x)|k, in\rangle = \langle o|A_\mu^{(in)}(x)|k, in\rangle - M\partial_\mu \langle o|\Lambda(x)|k, in\rangle, \tag{106}$$

which is identical to (83) in the Fermi gauge ($a = 1$).

By a simple calculation we find by comparison with (90) that

$$2M = \overline{\Pi}'(o) = \int_o^\infty d\lambda \frac{\Pi(\lambda)}{\lambda^2}, \tag{107}$$

which also follows in the conventional theory of the Fermi gauge[2].

We are now in a position to calculate the spectral functions in the lowest order. The result is

$$\Pi^{(o)}(p^2) = \frac{e^2}{12\pi^2}\left(1 + \frac{2m^2}{p^2}\right)\sqrt{1 - \frac{4m^2}{p^2}}\,\Theta(p^2 - 4m^2), \tag{108}$$

$$\Sigma_1^{(o)}(p^2) = \frac{me^2}{16\pi^2}\left(1 - \frac{m^2}{p^2}\right)\left(3 - a\frac{m^2}{p^2}\right)\Theta(p^2 - m^2), \tag{109}$$

$$\Sigma_2^{(o)}(p^2) = a\frac{e^2}{16\pi^2}\left(1 - \left(\frac{m^2}{p^2}\right)^2\right)\Theta(p^2 - m^2). \tag{110}$$

[1] The asymptotic condition (104) has been found by Rollnik et al., Z. f. Phys. **159** (1960) 482, for the case of the Fermi gauge. The author is grateful to G. Källén for calling his attention to this work and for pointing out that there may be some formal difficulties with this asymptotic condition.

[2] In order to find (107) one should use the equation

$$\langle o|[A_\mu - A_\mu^{(in)} + M\partial_\mu \Lambda, A_\nu - A_\nu^{(in)} + M\partial_\nu \Lambda]|o\rangle = \langle o|[A_\mu - A_\mu^{(in)}, A_\nu - A_\nu^{(in)}]|o\rangle,$$

and the derivation now proceeds exactly as in G. Källén, ibid., p. 350.

For $a = 1$ these reduce to the usual Fermi gauge spectral functions. It is seen that the wave function renormalization constant $1/N^2$ is not ultraviolet divergent for $a = 0$ (the Landau gauge) and not infrared divergent for $a = 3$ (the Yennie gauge) in the lowest order. This fact has been known for some time[1]. The self-mass presents in the lowest order a special problem, which we shall discuss in the next section.

11. The gauge dependence of the self-mass of the electron

We have seen in section 7 that the bare mass of the electron is not supposed to change under gauge transformations. As the physical electron mass obviously must be gauge independent we can immediately conclude that the self-mass δm must be gauge independent.

If, however, we calculate the difference in self-mass between an arbitrary gauge and the Fermi gauge in the lowest order we find by (109)

$$\delta m^{(o)} - \delta m_F^{(o)} = \bar{\Sigma}_1^{(o)}(m^2) - \bar{\Sigma}_{1F}^{(o)}(m^2) \tag{111}$$

$$= \int_{m^2}^{\infty} d\lambda \frac{\Sigma_1^{(o)}(\lambda) - \Sigma_{1F}^{(o)}(\lambda)}{\lambda - m^2} \tag{112}$$

$$= (1 - a)\frac{me^2}{16\,\pi^2}, \tag{113}$$

where we have denoted quantities from the Fermi gauge ($a = 1$) with a subscript F.

Unless we can find an error in our derivation, this result shows that there is an inconsistency in the theory. The error lies, however, in the step from (111) to (112), because (111) is the difference between two infinite numbers, the value of which depends on the method we prescribe for the calculation of this difference. The result shows that the prescription (112) is not correct and we must now try to find a better way of evaluating the difference.

If we introduce a cut-off in the photon propagator the formerly infinite numbers will become finite. Hence, we get

[1] B. Zumino, Journ. Math. Phys. **1** (1960) 1.

$$D_{\mu\nu}(k) = -i\frac{g_{\mu\nu}}{k^2} + i(1-a)\frac{k_\mu k_\nu}{(k^2)^2} \rightarrow \left(-i\frac{g_{\mu\nu}}{k^2} + i(1-a)\frac{k_\mu k_\nu}{(k^2)^2}\right)\frac{-\lambda^2}{k^2-\lambda^2} =$$

$$= -i\left(\frac{1}{k^2} - \frac{1}{k^2-\lambda^2}\right)g_{\mu\nu} + i(1-a)k_\mu k_\nu \cdot \left(\frac{1}{(k^2)^2} + \frac{1}{\lambda^2}\left(\frac{1}{k^2} - \frac{1}{k^2-\lambda^2}\right)\right)$$

where λ^2 is the cut-off parameter, which is supposed to be gauge independent. In the Fourier transform of the commutator and anti-commutator the expression

$$g_{\mu\nu}\delta(k^2) + (1-a)k_\mu k_\nu \delta'(k^2)$$

must then be replaced by

$$g_{\mu\nu}(\delta(k^2) - \delta(k^2-\lambda^2)) + (1-a)k_\mu k_\nu \cdot \left(\delta'(k^2) - \frac{1}{\lambda^2}(\delta(k^2) - \delta(k^2-\lambda^2))\right).$$

This expression is now used in the calculation of the spectral functions and we find the difference

$$\left.\begin{aligned} \Sigma_1^{(o)}(p^2) - \Sigma_{1F}^{(o)}(p^2) &= \frac{me^2}{16\pi^2}(1-a)\left(1-\frac{m^2}{p^2}\right)\frac{m^2}{p^2}\Theta(p^2-m^2) - \\ &- \frac{me^2}{32\pi^2}(1-a)\frac{p^2-m^2}{\lambda^2}\left(\left(1-\frac{m^2}{p^2}\right)^2\Theta(p^2-m^2) - \right. \\ &\left. - \left(1+\frac{\lambda^2-m^2}{p^2}\right)\sqrt{\frac{\lambda(p^2,m^2,\lambda^2)}{p^2}}\,\Theta(p^2-(m+\lambda)^2)\right). \end{aligned}\right\} \quad (114)$$

Here[1]

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$

is a quadratic form.

Remark that the last terms in (114) vanish for $\lambda^2 \rightarrow \infty$. If we insert (114) in (112), the integration can be performed and the result is identically zero. The original result (113) is exactly cancelled by a contribution from the last terms in (114). One should notice that the step from (111) to (112) now is perfectly allowed because both numbers in (111) are finite. Thus with this prescription the self-mass is gauge independent also in the lowest order.

[1] G. Källén, Elementary Particle Physics (Addison-Wesley, 1964).

The considerations in this section are a nice illustration of how carefully one must treat the infinite numbers met in canonical field theories of this kind.

12. Conclusions

It appears as if quantum electrodynamics in the covariant gauges of the type studied in this work is as consistent as the conventional theory of the Fermi gauge. But it is also clear that the formulation in that gauge is the simplest, not only because the photon propagator has its simplest form here, but also because the energy is not diagonalizable in any other gauge than the Fermi gauge. The transformations which connect different covariant gauges are of a rather singular nature. Although it migh be conceivable that quantum electrodynamics would only be consistent for one choice of gauge parameter, no special reasons have as yet been found which would support this possibility. One might argue that the apparently gauge dependent self-mass could be an indication of an inconsistency of quantum electrodynamics, but it is clear that this inconsistency only arises because the perturbation calculation gives rise to a divergent self-mass, and it therefore belongs to the general class of defects of the theory which are circumvented by the renormalization procedure.

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APPENDIX

The distribution

$$E(x) = 2\pi i \int dp\, \varepsilon(p) \delta'(p^2) e^{-ipx} \tag{A. 1}$$

has obviously the properties

$$\Box E(x) = D(x),$$

$$E(-x) = -E(x),$$

$$E(x) = \frac{\partial}{\partial \mu^2} \Delta(x, \mu^2) \Big|_{\mu^2 = o}$$

where $\Delta(x, \mu^2)$ is the well-known singular function

$$\Delta(x, \mu^2) = -2\pi i \int dp\, \varepsilon(p) \delta(p^2 - \mu^2) e^{-ipx}. \tag{A. 2}$$

By integration over p_0 and over angles we find from (A. 1)

$$E(x) = -\int \frac{d\bar{p}}{(2\pi)^3} \frac{e^{i\bar{p}\cdot\bar{x}}}{2\omega} \frac{\partial}{\partial \omega} \left(\frac{\sin \omega x_o}{\omega} \right) \tag{A. 3}$$

$$= \frac{1}{4\pi^2} \int_o^\infty d\omega \cos \omega |\bar{x}| \frac{\sin \omega x_o}{\omega}. \tag{A. 4}$$

From this we immediately get

$$E(x) = \frac{\varepsilon(x)}{8\pi} \Theta(x^2), \tag{A. 5}$$

and from (A. 3)

$$E(\bar{x}, o) = \dot{E}(\bar{x}, o) = \ddot{E}(\bar{x}, o) = 0; \; \dddot{E}(\bar{x}, o) = \delta(\bar{x}).$$

From (A. 4) it is clear that the positive frequency part of $E(x)$ is divergent for $\omega \rightarrow 0$. But it also seems as if the positive frequency part of $\partial_\mu E(x)$

were convergent. Let us therefore study this divergence somewhat closer. We use here the well-known expansions of the singular functions

$$\Delta(x,\mu^2) = -\frac{1}{2\pi}\varepsilon(x)\delta(x^2) + \frac{\mu^2}{8\pi}\varepsilon(x)\Theta(x^2)\left(1-\frac{\mu^2}{8}x^2\right) + O(\mu^4),$$

$$\Delta^{(1)}(x,\mu^2) = -\frac{1}{2\pi^2}\frac{1}{x^2} + \frac{\mu^2}{4\pi^2}\log\frac{\gamma\mu|x|}{2} - \frac{\mu^2}{8\pi^2} +$$

$$+\frac{\mu^4x^2}{32\pi^2}\left(\frac{5}{4}-\log\frac{\gamma\mu|x|}{2}\right) + O(\mu^4).$$

Equation (A. 5) is easily seen to follow from the first equation by differentiation after μ^2. By differentiation of the second we get

$$\frac{\partial\Delta^{(1)}(x,\mu^2)}{\partial\mu^2} = \frac{1}{4\pi^2}\log\frac{\gamma\mu|x|}{2} + \frac{\mu^2x^2}{16\pi^2}\left(\frac{5}{4}-\log\frac{\gamma\mu|x|}{2}\right) - \frac{\mu^2x^2}{64\pi^2} + O(\mu^2),$$

and this is clearly not convergent for $\mu^2 \to 0$. But the gradient of this expression is convergent in the limit

$$\partial_\lambda\frac{\partial\Delta^{(1)}(x,\mu^2)}{\partial\mu^2} = \frac{\partial}{\partial\mu^2}\partial_\lambda\Delta^{(1)}(x,\mu^2) \to \frac{1}{4\pi^2}\frac{x_\lambda}{x^2}.$$

This means that we may define the distributions

$$\partial_\lambda E^{(1)}(x) = \frac{\partial}{\partial\mu^2}\partial_\lambda\Delta^{(1)}(x,\mu^2)\Big|_{\mu^2=o},$$

$$\partial_\lambda E^{(\pm)}(x) = \frac{\partial}{\partial\mu^2}\partial_\lambda\Delta^{(\pm)}(x,\mu^2)\Big|_{\mu^2=o},$$

while the distributions $E^{(1)}$ and $E^{(\pm)}$ do not exist.

From (A. 3) it now easily follows that

$$E(x) = \frac{1}{2\Delta}\left(D(x) - x_o\dot{D}(x)\right).$$

Likewise it follows from

$$\partial_\mu E^{(1)}(x) = 2\pi i\int dk k_\mu\delta'(k^2)e^{-ik\cdot x}$$

that
$$\partial_\mu E^{(1)}(x) = \frac{1}{2\Delta}\,\partial_\mu(D^{(1)}(x) - x_o\dot{D}^{(1)}(x)), \tag{A. 6}$$

where $\frac{1}{2\Delta}$ and ∂_μ *cannot* be interchanged.

In order to find the propagator

$$D_{\mu\nu}(x-y) = \tfrac{1}{2}\langle o|\{A_\mu(x),A_\nu(y)\}|o\rangle + \tfrac{1}{2}\,\varepsilon(x-y)\langle o|[A_\mu(x),A_\nu(y)]|o\rangle \tag{A. 7}$$

we shall first calculate the vacuum expectation value of the anti-commutator. We express A_μ in terms of the Fermi field through eq. (36). Using the fact that

$$\langle o|\{A_\mu^F(x),A_\nu^F(y)\}|o\rangle = -g_{\mu\nu}D^{(1)}(x-y)$$

we get by means of (37)

$$\langle o|\{\Lambda(x),A_\nu^F(y)\}|o\rangle = -\partial_\nu^x D^{(1)}(x-y),$$

$$\langle o|\{\Lambda(x),\Lambda(y)\}|o\rangle = 0.$$

Then from these equations and eqs. (35) and (A. 6) we finally get

$$\langle o|\{A_\mu(x),A_\nu(y)\}|o\rangle = -g_{\mu\nu}D^{(1)}(x-y) - (1-a)\partial_\mu\partial_\nu E^{(1)}(x-y).$$

From the properties of $E(x)$ it follows that

$$\varepsilon(x-y)\partial_\mu\partial_\nu E(x-y) = \partial_\mu\partial_\nu(\varepsilon(x-y)E(x-y)),$$

so that we may write the propagator in the form

$$D_{\mu\nu}(x-y) = -g_{\mu\nu}\left(\tfrac{1}{2}\,D^{(1)}(x-y) + \frac{i}{2}\,\varepsilon(x-y)D(x-y)\right) -$$
$$-(1-a)\frac{\partial}{\partial\mu^2}\,\partial_\mu\partial_\nu\left(\tfrac{1}{2}\,\Delta^{(1)}(x-y,\mu^2) + \frac{i}{2}\,\varepsilon(x-y)\Delta(x-y,\mu^2)\right)\Bigg|_{\mu^2=o}.$$

In momentum space we find

$$D_{\mu\nu}(k) = -g_{\mu\nu}\frac{i}{k^2+i\varepsilon} + (1-a)\frac{\partial}{\partial\mu^2}\,k_\mu k_\nu\,\frac{i}{k^2-\mu^2+i\varepsilon}\Bigg|_{\mu^2=o} =$$
$$= -i\frac{g_{\mu\nu}}{k^2+i\varepsilon} + i(1-a)\frac{k_\mu k_\nu}{(k^2+i\varepsilon)^2}.$$

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